Closing Thu: 12.4(1)(2), 12.5(1)
Closing next Tue: 12.5(2)(3), 12.6
Closing next Thu: 13.1, 13.2

12.5 Lines and Planes in 3D

Lines: We use parametric equations for 3D lines. Here's a 2D warm-up:

Consider the line: $y = 4x + 5$.

(a) Find a vector parallel to the line. Call it **v.**

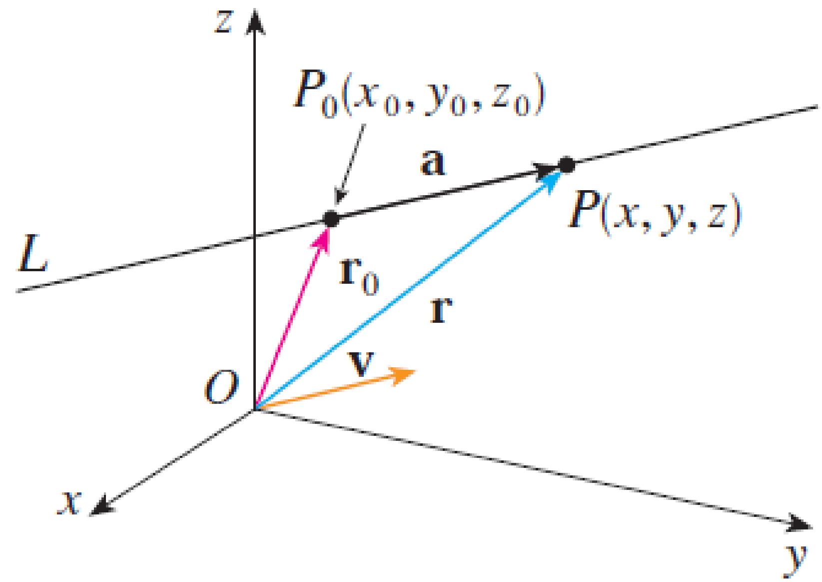
(b) Find a vector whose head touches the line when drawn from the origin. Call it $\mathbf{r_0}$.

(c) Observe, we can reach all other points on the line by walking along $\mathbf{r_0}$, then adding scale multiples of **v**.

This same idea works to describe any line in 2- or 3-dimensions.

Summary of Line Equations

Let (x,y,z) be any point on the line and $\boldsymbol{r} =< x, y, z >$ = a vector pointing to this point from the origin.



Find **<u>a direction vector</u>** and **<u>a point</u>** on the line.

1. $\boldsymbol{v} = \langle a, b, c \rangle$ *direction vector*

2. $\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ *position vector*

$\boldsymbol{r} = \boldsymbol{r_0} + \mathrm{t}\,\boldsymbol{v}$	*vector form*
$(x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct)$ $x = x_0 + at,$ $y = y_0 + bt,$ $z = z_0 + ct.$	*parametric form*
$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$	*symmetric form*

Basic Example – Given Two Points:
Find parametric equations of the line thru P(3, 0, 2) and Q(-1, 2, 7).

General Line Facts

1. Two lines are **parallel** if their direction vectors are parallel.

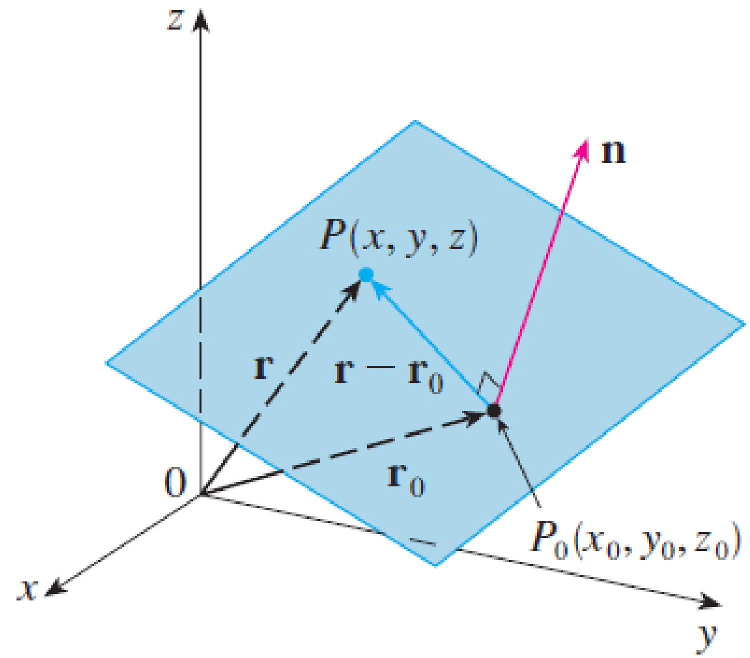
2. Two lines **intersect** if they have an (x,y,z) point in common.
   Use different parameters when you combine!

   Note: The *acute angle of intersection* is the acute angle between the direction vectors.

3. Two lines are **skew** if they don't intersect and aren't parallel.

Summary of Plane Equations

Let (x,y,z) be any point on the plane and $\boldsymbol{r} = < x, y, z >$ = a vector pointing to this point from the origin.



Find **a normal vector** and **a point** on the plane.

1. $\boldsymbol{n} = \langle a, b, c \rangle$ *normal vector*

2. $\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ *position vector*

$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$ *vector form*

$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ *standard form*

If you expand out standard form you can write:

$ax - ax_0 + by - by_0 + cz - cz_0 = 0$

$ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$

Basic Example – Given Three Points:
Find the equation for the plane through the points P(0, 1, 0), Q(3, 1, 4), and R(-1, 0, 0)

General Plane Facts

1. Two planes are **parallel** if their normal vectors are parallel.

2. If two planes are not parallel, then they must intersect to form a line.

   2a. The *acute angle of intersection* is the acute angle between their normal vectors.

   2b. The planes are orthogonal if their normal vectors are orthogonal.

12.5 Summary

Lines: Find a POINT and DIRECTION.

$\boldsymbol{v} = \langle a, b, c \rangle$ *direction vector*

$\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ *position vector*

$x = x_0 + at, y = y_0 + bt, z = z_0 + ct.$

Planes: Find a POINT and NORMAL

$\boldsymbol{n} = \langle a, b, c \rangle$ ***normal*** *vector*

$\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ *position vector*

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

To find equations for a line

- Info given? → Done.
- Find two points → $\vec{v} = \overrightarrow{AB}$ (subtract components); $\overrightarrow{\boldsymbol{r_0}} = \vec{A}$

To find the equation for a plane

- Info given? → Done.
- Find three points → Two vectors parallel to the plane: $\overrightarrow{AB}$ and $\overrightarrow{AC}$ → $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$; $\overrightarrow{\boldsymbol{r_0}} = \vec{A}$

lines parallel – directions parallel.
lines intersect – make (x,y,z) all equal (different param!)
Otherwise, we say they are skew.

planes parallel – normals parallel.
Otherwise, the planes intersect.

1. Find an equation for the line that goes through the two points $A(1,0,-2)$ and $B(4,-2,3)$.

2. Find an equation for the line that is parallel to the line $x = 3 - t$, $y = 6t, z = 7t + 2$ and goes through the point $P(0,1,2)$.

3. Find an equation for the line that is orthogonal to $3x - y + 2z = 10$ and goes through the point $P(1,4,-2)$.

4. Find an equation for the line of intersection of the planes

$5x + y + z = 4$ and

$10x + y - z = 6.$

1. Find the equation of the plane that goes through the three points A(0,3,4), B(1,2,0), and C(-1,6,4).

2. Find the equation of the plane that is orthogonal to the line $x = 4 + t, y = 1 - 2t, z = 8t$ and goes through the point P(3,2,1).

3. Find the equation of the plane that is parallel to $5x - 3y + 2z = 6$ and goes through the point P(4,-1,2).

4. Find the equation of the plane that contains the intersecting lines $x = 4 + t_1, y = 2t_1, z = 1 - 3t_1$ and $x = 4 - 3t_2, y = 3t_2, z = 1 + 2t_2$.

5. Find the equation of the plane that is orthogonal to $3x + 2y - z = 4$ and goes through the points P(1,2,4) and Q(-1,3,2).

1. Find the intersection of the line $x = 3t$, $y = 1 + 2t$, $z = 2 - t$ and the plane $2x + 3y - z = 4$.

2. Find the intersection of the two lines $x = 1 + 2t_1$, $y = 3t_1$, $z = 5t_1$ and $x = 6 - t_2$, $y = 2 + 4t_2$, $z = 3 + 7t_2$ (or explain why they don't intersect).

3. Find the intersection of the line

$x = 2t$, $y = 3t$, $z = -2t$ and the sphere

$x^2 + y^2 + z^2 = 16$.

4. Describe the intersection of the

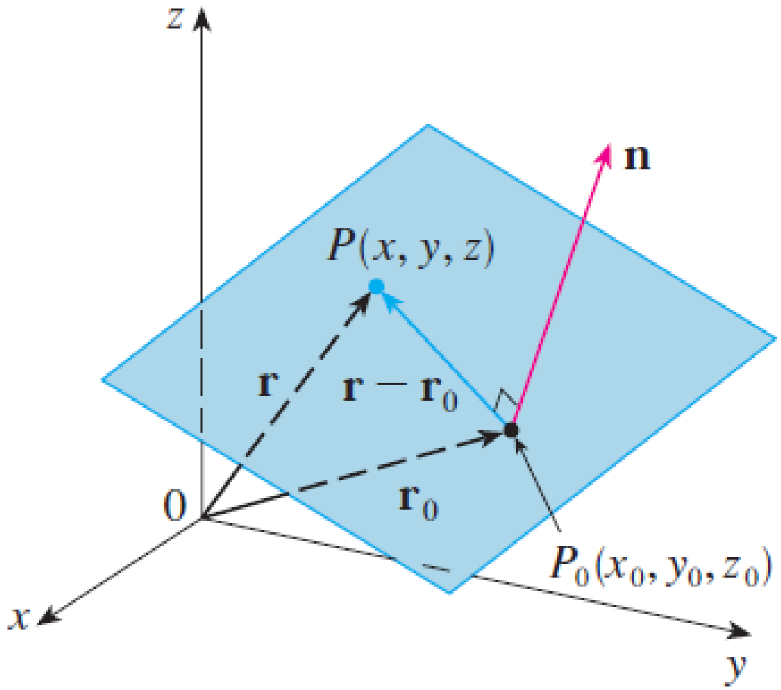
plane $3y + z = 0$ and the sphere

$x^2 + y^2 + z^2 = 4$.

Questions directly from old tests:

1. Consider the line thru (0, 3, 5) that is orthogonal to the plane $2x - y + z = 20.$ Find the point of intersection of the line and the plane.

2. Find the equation for the plane that contains the line $x = t, y = 1 - 2t, z = 4$ and the point (3,-1,5).



Side comment
(one of the many uses of projections)

If you want the distance between two *parallel* planes, then

(a) Find *any* point on the first plane (x_0, y_0, z_0) and *any* point on the second plane (x_1, y_1, z_1).

(b) Write $\mathbf{u} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

(c) Project $\mathbf{u}$ onto one of the normal vector $\mathbf{n}$.

$|\text{comp}_{\mathbf{n}}(\mathbf{u})|$ = dist. between planes